Self-Mass and Equivalence in Special Relativity

By J. KALCKAR, J. LINDHARD *and* O. ULFBECK

Det Kongelige Danske Videnskabernes Selskab
Matematisk-fysiske Meddelelser **40**: 11

Kommissionær: Munksgaard
København 1982

Contents

Page

§ 1. Introduction 3
Basic equivalence. The classical electron model.

§ 2. Self-Mass and Self-Force of Accelerated Systems 7
Mass determined from forces and acceleration. Electrostatic self-mass. Basic equations motion. Shortcomings of standard formulae. Self-mass for point force.

§ 3. Systematic Description by Means of Accelerated Reference Frame 15
Basic properties of Møller box. Electrostatic interactions in the Møller box. Equivalence for kinetic energy.

§ 4. Equivalence for Atomic Binding Energies 21
Classical hydrogen atom. Quantal hydrogen atom. The Dirac equation. Thomas-Fermi atom.

§ 5. Conclusions and General Outlook 29

Appendix A. Electrodynamics in Møller Box 33

Appendix B. Static Potentials in Møller Box 36

Appendix C. Klein-Gordon Equation in Møller Box 39

References 42

Synopsis

We study dynamical aspects of equivalence between mass and energy, for systems of interacting particles. The starting-point consists in the classical formulae for electromagnetic self-momentum, self-energy and self-force. These formulae possess puzzling terms which have been subject to various explanations, like compensating Poincare stresses, or were bypassed through attempts of redefinition of the classical electron model. By means of a comprehensive study of acceleration processes we show that there is a crucial error in the usual derivations of self-force. We derive a basic acceleration equation for a point-like system, with detailed equivalence. It also follows that the standard formulae for self-energy and self-momentum are, at best, misleading. Next, we study how systems are to be described in an accelerated, rigid coordinate frame – the Møller box. In considerable detail we investigate classical and quantal equations of motion for fields and for particles in the Møller box, including the Dirac equation for the hydrogen atom, arriving at equivalence. Finally we discuss properties of composite systems as compared with properties of particles.

JØRGEN KALCKAR, OLE ULFBECK
Niels Bohr Institute
University of Copenhagen

JENS LINDHARD
Institute of Physics
University of Aarhus
Denmark

Printed in Denmark by Bianco Lunos Bogtrykkeri A/S, ISSN 0023-3323. ISBN 87-7304-124-6

§1. Introduction

The present paper contains a study of equivalence between mass and energy, for a system of interacting particles. This question may appear trivial from the point of view of general principles in special relativity, for there can be no doubt about the validity of equivalence as a basic statement. But if we turn to actual calculations on systems with Coulomb interaction, we find that the resulting self-energies and self-momenta do not correspond to equivalence, and lack four-vector properties[1]. This is certainly surprising to the uninitiated, for one would expect beforehand that the Maxwell equations must give equivalence unambiguously and in a straightforward manner, since special relativity, in a sense, is suspended in the Maxwell equations. It should be added that the results in question have often been treated in connection with the problem of electron self-energies where they apparently required the presence of non-electromagnetic forces. Because of the complications, some authors have preferred to *define* an electromagnetic energy-momentum four-vector for the electron[1,9]. This will hardly do, however. The basic classical case is not an electron, but a macroscopic system, for which one is not free to define the electromagnetic self-energy or self-momentum.

These preliminary comments will be enlarged upon in the remainder of this chapter. But they indicate the aim of the present paper. In fact, we hope to convince the reader that simple acceleration processes, if studied with care, reveal that there is not only equivalence, but even *detailed* equivalence: each individual term of, e.g., the interaction energy, has separate equivalence. The basic conclusions in this respect are contained in §2, where the connection between mass, forces and acceleration is studied. A central issue is the question of comparison of forces acting in different points of a system. In §3 is presented the more systematic treatment of accelerated frames of reference. Next, in §4, we calculate the various contributions to self-mass in a number of classical and quantal cases, including the Dirac equation for a hydrogen atom.

As outlined, our study has an immediate background. But it is also a necessary step in a more general pursuit: the endeavour to understand composite systems

and elementary particles, including the connection between them and between their classical and quantal descriptions. At the end of this paper, in § 5, we outline general viewpoints on this matter.

Basic equivalence

The equivalence between energy and inertial mass was first established by Einstein[2] (cf. also v. Laue[3]). He considered the change suffered by a system emitting electromagnetic radiation. The usual derivation consists in showing, first, that the energy E and momentum **P** of any closed system constitute a four-vector. The basis for this result is the special principle of relativity, combined with conservation of energy and momentum for initial and final states of a collision process (corresponding to Einstein's idealized experiments). Second, the four-vector may be written as

$$E = E'\gamma, \tag{1.1}$$

$$\mathbf{P} = \frac{E'}{c^2}\gamma\mathbf{v}, \tag{1.2}$$

where E' is a constant, and

$$\gamma = \frac{1}{(1-\mathbf{v}^2/c^2)^{1/2}}\,.$$

Now, on the one hand, E' in (1.1) has to be the energy of the system in the rest frame. On the other hand, E'/c^2 in (1.2) must be the mass M of the system, belonging to the non-relativistic limit $v \ll c$, and so we obtain equivalence,

$$M = \frac{E'}{c^2}. \tag{1.3}$$

Equivalence is therefore derived by comparing initial and final states of an elastic or inelastic process. The proof concerns not only a stable system; it includes unstable systems. Although it appears that the proof of equivalence is concerned with only the total energy of a system, still there are evidently cases where part of the energy must have equivalence. Moreover, one can divide the total energy of a system into well-defined average contributions from various forms of energy, as exemplified by the virial theorem. Beforehand, one would expect individual equivalence from these clearly separated contributions. Thus, it is natural to investigate the possible validity of *detailed* equivalence, as formulated in the preamble.

The previous results may be put on a more comprehensive form if we introduce the Lagrangian of the system. In fact, the above momentum-energy four-vector

with its derivation from collisions, must be connected to a variational principle, albeit with limited validity. In an inertial frame where the system has velocity $\mathbf{v}$, the corresponding Lagrangian must be

$$L = -E'\left(1-\frac{\mathbf{v}^2}{c^2}\right)^{1/2}. \tag{1.4}$$

In this formula the internal energy E' is a constant of the motion for a given internal state of the system. If we now observe the given system from another frame, where it has velocity $\mathbf{v}+\delta\mathbf{v}$, the change δL becomes

$$\delta L = \delta\mathbf{v}\cdot\mathbf{P}, \tag{1.5}$$

where $\mathbf{P}$ is given by (1.2). Furthermore, the quantity $E = -L+\mathbf{v}\cdot\mathbf{P}$ corresponds to (1.1).

In (1.5) we are concerned with a variation where the internal state of the system is kept unchanged. Thus, we have obtained equivalence, $E' = Mc^2$, by being able to separate the external velocity variable, $\mathbf{v}$ in (1.4), from the internal variables of the system, concealed in the constant E'. Moreover, for soft collisions – if the internal state of the system is not changed during a collision – the equation of motion will be based on (1.4), i.e. the kinetic contribution to the total Lagrangian.

In itself, eq. (1.4) reasserts the surmises about detailed equivalence made above. Thus, when the system is in internal statistical equilibrium, and E' separates into definite terms according to the virial theorem, these terms should contribute separate mass terms in the account of the system.

The classical electron model

Already before the advent of special relativity, Poynting's theorem of density of field energy was utilized in several calculations of self-energy and self-momentum of a charged body (cf. e.g. Jammer[4]). The foremost contribution was made by Lorentz[5]. The results were hardly changed at all by special relativity. We shall illustrate the situation in terms of the so-called classical electron model, as quoted in numerous monographs (e.g., Jackson[1], Pais[6], Feynman[7]).

Consider then a stable spherical shell with radius a, on which a total charge Q is uniformly distributed. In a inertial frame K the shell moves with velocity $\mathbf{v}$. According to standard results, the densities of momentum and energy of an electromagnetic field are given by the field strengths

$$\mathbf{g}(\mathbf{r},t) = \frac{1}{4\pi c}\mathbf{E}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t), \tag{1.6}$$

$$u(\mathbf{r},t) = \frac{1}{8\pi}(\mathbf{E}^2(\mathbf{r},t) + \mathbf{B}^2(\mathbf{r},t)). \tag{1.7}$$

In these equations we introduce the field belonging to the shell, as observed in the frame K. By integrating over all space we find a momentum $\mathbf{P}_{el}$, and an energy E_{el} (cf. Jackson[1], Becker and Sauter[8], Rohrlich[9])

$$\mathbf{P}_{el} = \int d^3r\ \mathbf{g}(\mathbf{r},t) = \frac{4}{3}\frac{Q^2}{2a}\gamma\mathbf{v}, \tag{1.8}$$

$$E_{el} = \int d^3r\ u(\mathbf{r},t) = \frac{Q^2}{2a}\gamma\left(1+\frac{1}{3}\frac{\mathbf{v}^2}{c^2}\right). \tag{1.9}$$

In particular, it is seen that, in the rest frame, the momentum is $\mathbf{P}'_{el} = 0$, and the energy $E'_{el} = Q^2/2a$. Therefore it follows that not only is equivalence lacking, but also $\mathbf{P}_{el}$, E_{el}, $\mathbf{P}'_{el}$, and E'_{el} fail to transform like the four-vector (1.1), (1.2). As is well known, this curious result cannot be rejected out of hand, the reason being that $\mathbf{P}_{el}$ and E_{el} do no represent the *total* momentum and energy of the shell. The shell in question must in any case be stabilized by other, nonelectromagnetic, forces. These forces, or Poincaré stresses, should then compensate the erratic behaviour of (1.8), (1.9), giving a correct total four-vector.

The results (1.8) and 1.9) are obtained somewhat indirectly, in a sense. Their basis, i.e. the densities (1.6) and (1.7), was derived in turn by studying the action of forces from material charges on an electromagnetic field. Therefore, by omitting the intermediate step (1.6), there should be a more direct, but apparently equivalent, way of obtaining the self-mass due to Coulomb interaction. In fact, one can instead find the electromagnetic self-force of an accelerated shell of charge. An early calculation of this kind was performed by Born[10] (cf. also Heitler[11], and Jackson[1]). If the shell is momentarily at rest, but accelerated, at time $t = 0$, one may find the electric field $\mathbf{E}_s(\mathbf{r},t)$ caused by it. We assume that the acceleration is small, or $ga/c^2 \ll 1$. It follows that $\mathbf{E}_s(\mathbf{r},t)$ is linear in $\mathbf{g}$. Let further $\varrho(\mathbf{r})$ denote the internal charge distribution of the shell. The total self-force $\mathbf{F}_s$ is then linear in the acceleration

$$\mathbf{F}_s = \int d^3r\,\varrho(\mathbf{r})\,\mathbf{E}_s(\mathbf{r},t=0)\ . \tag{1.10}$$

By these means the self-mass was calculated as the ratio F_s/g, the result being in agreement with (1.8).

It thus looks as if the previous result (1.6) has been vindicated in an elementary way by the self-force (1.10). The latter becomes our starting-point, however. For although it concerns non-relativistic motions and an apparently innocent acceleration process, still this process contains unexpected relativistic pitfalls, and (1.10) is not connected to the self-mass, as we shall see ind § 2.

§2. Self-Mass and Self-Force of Accelerated Systems

In this chapter we attempt to find the way in which inertial mass can be determined by means of acceleration processes. Since we know that there may be hidden difficulties in this problem, we try to be careful – and thereby perhaps overly cautious – in deriving the relativistic connection between acceleration, forces, and mass. We do it in two steps. First, we look for the physically simplest acceleration process for a system of finite size. Next, we find the expression for the mass of a system, given in terms of its acceleration and the forces acting on it. We will then be ready to find actual self-masses for charged systems, and have also prepared the way for the more systematic treatment in the following chapters.

The problem at hand can be exemplified by an elastic body originally at rest, and in equilibrium, in an inertial frame. We want to transfer it to another inertial frame, where it should finally be at rest and in the same state of equilibrium as before. The simplest way in which to bring about this change is to have an adapted acceleration of the various parts of the system, such that it is moved as if it were rigid. In fact, by means of the idealized process of rigid acceleration we avoid producing internal stress or excitations in the system, as well as growing deformations. It would of course be possible to employ acceleration processes other than the rigid one; they would be more complicated, however, and would need the rigid acceleration as a standard of reference.

Rigid acceleration

As our first step we therefore consider the kinematical consequences of rigid acceleration of a static system. Then there exist successive frames in which the velocities of all constituent particles vanish simultaneously. Consequently the time, t_i, at which the i'th particle obtains a given velocity, $\mathbf{v}_0$, in the inertial

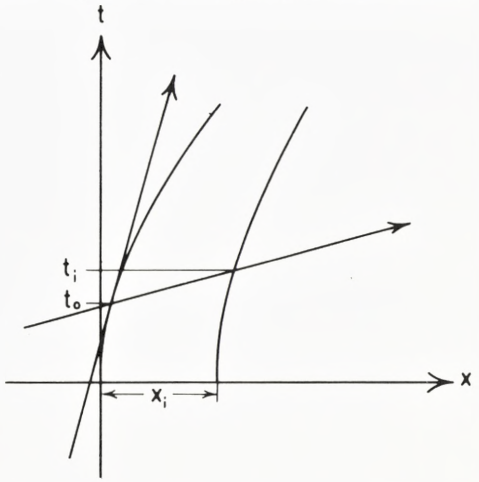

Fig. 1. Space-time diagram showing world lines of system of accelerated particles. In the inertial frame K the time t_i at which the i'th particle obtains a given velocity, $\mathbf{v}_0$, depends on its position x_i relative to the reference point. The instances t_i and t_0 are simultaneous in the rest frame, i.e. the oblique line corresponds to simultaneity in this frame.

frame K depends on its position, $\mathbf{r}_i$ (see Fig. 1). For constant acceleration, $\mathbf{g}_i$, of the i'the particle we then have, since we consider small time intervals and velocities,

$$\mathbf{v}_0 = \mathbf{g}_i t_i \,. \tag{2.1}$$

Let us measure positions relative to some reference point chosen arbitrarily within the system, and let t_0 denote the time during which the reference point has been accelerated with acceleration $\mathbf{g}_0$. Since the instances t_0 and t_i in the frame K must correspond to simultaneity in the rest frame, they are to first order in v/c related by the equation

$$t_i = t_0 + \frac{\mathbf{v}_0 \cdot \mathbf{r}_i}{c^2} = t_0\left(1 + \frac{\mathbf{g}_0 \cdot \mathbf{r}_i}{c^2}\right). \tag{2.2}$$

It therefore follows that the accelerations of the various points $\mathbf{r}_i$ must obey the relation

$$\mathbf{g}_i = \frac{\mathbf{g}_0}{1 + \mathbf{g}_0 \cdot \mathbf{r}_i / c^2} \,. \tag{2.3}$$

Thus, the acceleration of the points $\mathbf{r}_i$ decreases in the direction of $\mathbf{g}_0$. This effect exactly corresponds to the Lorentz contraction of the system as measured from the frame K.

Mass determined from forces and acceleration

As a second step, let us study the connection between force, acceleration, and mass. The fundamental relation between the three is obtained in the idealized case of a point particle. In fact, consider a point particle at rest, and with mass m. If it acquires a small acceleration $\mathbf{g}$, the applied force must be $\mathbf{F} = \mathbf{g}m$. Moreover, during a time δt, its change of momentum and velocity are, respectively, $\delta\mathbf{p} = \mathbf{F}\delta t$ and $\delta\mathbf{v} = \mathbf{g}\delta t$. This result has an immediate consequence. For suppose that, by applying the above force $\mathbf{F}$ in one point of a composite system, we obtain the acceleration $\mathbf{g}$ of this point, while the system remains, internally, in a stationary state. Since the momentum transfer and velocity change remain as before, the composite system must have the same mass m as the above particle. Presumably, part of its mass is then due to deformation energy caused by the acceleration. These seemingly trivial conclusions give one important clue to self-mass problems, as shown in an example at the end of this chapter.

Having verified the basic results belonging to a point force, we next consider acceleration of a system where forces are applied in several points. It follows from, e.g., eq. (2.3) that in special relativity there must be a somewhat intricate connection between forces on a system, its acceleration, and its total mass. Because of this, and because of the important consequences, we treat the problem at hand in an elementary and somewhat elaborate manner. We also want to show that

one is not concerned with new definitions, but instead with an inherent physical property of accelerated systems: although the velocities are non-relativistic, we sometimes have to introduce a relativistic correction to the usual conception of forces.

Consider then a set of point masses m_i, at $\mathbf{r}_i$, initially at rest and with no mutual forces. By means of suitable external forces we can accelerate the masses together, according to (2.3). We must act on them with the individual forces $\mathbf{F}_i$, located at $\mathbf{r}_i$,

$$\mathbf{F}_i = \mathbf{g}_0 \frac{m_i}{1 + \mathbf{g}_0 \cdot \mathbf{r}_i / c^2}, \tag{2.4}$$

where the acceleration at $\mathbf{r} = 0$ is $\mathbf{g}_0$. Now, if we compute the total force $\mathbf{F}$, we obtain

$$\mathbf{F} = \sum_i \mathbf{F}_i = \sum_i \mathbf{g}_0 \frac{m_i}{1 + \mathbf{g}_0 \cdot \mathbf{r}_i / c^2}, \tag{2.5}$$

which quantity is not proportional to the total rest mass, $M = \sum_i m_i$. Instead, the exact expression for the mass M is, by (2.4),

$$\mathbf{g}_0 M = \sum_i \mathbf{F}_i \left(1 + \frac{\mathbf{g}_0 \cdot \mathbf{r}_i}{c^2}\right). \tag{2.6}$$

In point of fact, we have here normalized all forces to the point $\mathbf{r} = 0$, with acceleration $\mathbf{g}_0$. At first, eq. (2.6) might appear to be an unnecessary elaboration, for if $\mathbf{g}_0$ is imagined to be sufficiently small, it looks as if the factors $(1 + \mathbf{g}_0 \cdot \mathbf{r}_i / c^2)$ can be replaced by unity. That will also be true in many cases, but for self-forces it is in error, because they contain large leading interaction terms, which would cancel if this replacement were made.

The conception of rigid acceleration may appear a little artificial for non-interacting point masses. But the idea is, as before, that we can replace this system by an actual system of interacting masses, e.g., an elastic body. In order not to deform the body more and more during the acceleration, we must keep to the prescribed rigid acceleration. As before, the total mass of the elastic body must be the same as that of the non-interacting point masses, if the accelerations and forces are the same. The formula (2.6) is therefore the general expression for the mass of the system, calculated in the simplest consistent situation.

Electromagnetic self-mass

Suppose that a system consists of charged particles, with individual masses m_i. At time $t = 0$ in the frame K the particles are all at rest with separations r_{ik} and electrostatic energy

$$U = \frac{1}{2} \sum_{i,k}^{i \neq k} \frac{q_i q_k}{r_{ik}} , \tag{2.7}$$

where q_i is the charge of the i'th particle. We assume that the particles are all accelerated according to (2.3), i.e. with rigid motion. By means of the Lienard-Wiechert potentials we can find, for $t = 0$, the total electric field at $\mathbf{r}_i$, as generated by all the other particles. It is readily shown to be[1,12], to first order in the accelerations $\mathbf{g}_k$,

$$\mathbf{E}(\mathbf{r}_i) = \sum_{k \neq i} q_k \left\{ \frac{\mathbf{r}_{ik}}{r_{ik}^3} - \frac{1}{2 r_{ik} c^2} \left(\mathbf{g}_k + \frac{(\mathbf{g}_k \cdot \mathbf{r}_{ik}) \mathbf{r}_{ik}}{r_{ik}^2} \right) \right\}, \tag{2.8}$$

where $\mathbf{r}_{ik} = \mathbf{r}_i - \mathbf{r}_k$, and $\mathbf{g}_k$ is given by (2.3). Note that the first term in the brackets is the dominating Coulomb force.

The self-forces are $q_i \mathbf{E}(\mathbf{r}_i)$, and by applying, in principle, external forces $\mathbf{F}_i = -q_i \mathbf{E}(\mathbf{r}_i) + m_i \mathbf{g}_i$, we maintain the stipulated acceleration. The total mass is now given by (2.6) and (2.8), i.e.

$$\mathbf{g}_0 M = \mathbf{g}_0 \sum_i m_i - \sum_{i,k}^{i \neq k} q_i q_k \left\{ \frac{\mathbf{r}_{ik}}{r_{ik}^3} - \frac{1}{2 r_{ik} c^2} \left(\mathbf{g}_k + \frac{(\mathbf{g}_k \cdot \mathbf{r}_{ik})}{r_{ik}^2} \mathbf{r}_{ik} \right) \right\} \left\{ 1 + \frac{\mathbf{g}_0 \cdot \mathbf{r}_k}{c^2} \right\}. \tag{2.9}$$

The right-hand side of (2.9) contains g_0 to first power, and higher powers. The latter terms are to be omitted, however, since we disregarded higher order terms in (2.8). But then the multiplying factor $1 + \mathbf{g}_0 \cdot \mathbf{r}_i / c^2$ is needed only for the dominating Coulomb term, and we may also put $\mathbf{g}_k = \mathbf{g}_0$ in the second term within the brackets. Finally, a cancellation occurs for all terms directed along $\mathbf{r}_{ik}$. We can then divide out the common factor $\mathbf{g}_0$, and obtain the inertial mass

$$M = \sum_i m_i + \frac{1}{2c^2} \sum_{i,k}^{i \neq k} \frac{q_i q_k}{r_{ik}} = \sum_i m_i + \frac{U}{c^2}. \tag{2.10}$$

Hereby we have found separate equivalence in a simple example of Coulomb interaction. Equivalence even applies for each individual pair of particles.

As expected, the calculation for the Coulomb interaction is independent of the presence of other compensating forces, whose contribution to self-mass may be obtained separately, if the corresponding field equations are known. But even if the compensating forces are only known on an approximate, non-relativistic form, we can obtain their contribution to the self-mass. At the same time, our basic equation (2.6) gives a condition to be fulfilled by the retarded solution of these forces, to first order in relative velocities.

We omitted the deformation energy caused by the acceleration, and its equivalence. This is mainly because the acceleration did not correspond to a normal physical situation. The external forces introduced were artificial, in that

they exactly took care of maintaining the configuration of the particles within the undisturbed system. In actual acceleration processes one can instead be concerned with an external electric field which is constant in space and time. Then, the configuration of the particles will be changed slightly from that of the undisturbed system. Hereby, deformation energy, and its equivalence, can be obtained. In many cases, such deformation terms are of higher order in g_0, and therefore do not affect the basic result (2.10). We shall presently discuss a simple example where deformation energy plays a major role.

Basic equation of motion

From the previous results it is easy to formulate the basic equation of motion of charged system momentarily at rest, and placed in a weak external electric field $\mathbf{E}_{ext}(\mathbf{r},t)$ varying slowly in space and time. By a slow variation in space we mean that the relative change of $\mathbf{E}_{ext}(\mathbf{r},t)$ is small within the system. We suppose that the field varies sufficiently slowly so that the system remains in a quasistationary state. Eq. (2.9) provides an expression for the acceleration $\mathbf{g}_0$ of a standard point $\mathbf{r}_0$, times the total mass M as arising from Coulomb interaction and from other energy contributions in the system. Next, according to (2.6) the product $\mathbf{g}_0 M$ is equal to the weighted sum of external forces,

$$\mathbf{g}_0 M = \sum_i F_i \left(1 + \frac{\mathbf{g}_0 \cdot (\mathbf{r}_i - \mathbf{r}_0)}{c^2}\right) = \sum_i q_i \mathbf{E}_{ext}(\mathbf{r}_i, t) \left(1 + \frac{\mathbf{g}_0 \cdot (\mathbf{r}_i - \mathbf{r}_0)}{c^2}\right). \quad (2.11)$$

Here, we expand $\mathbf{E}_{ext}$ in powers of $\mathbf{r}_i - \mathbf{r}_0$, and include only first order terms on the right hand side. But since the standard point $\mathbf{r}_0$ may be freely chosen, we place it at the charge centre, $\mathbf{r}_0 = \mathbf{r}_c \equiv \sum_i q_i \mathbf{r}_i / q$, where $q = \sum_i q_i$ is the total charge of the system. For it then turns out that first order terms in $\mathbf{r}_i - \mathbf{r}_0$ disappear, and we are left with $q\mathbf{E}_{ext}(\mathbf{r}_c, t)$ on the right hand side of (2.11). The equation of motion is now simply, in the momentary rest frame,

$$\mathbf{g}_c M = q\mathbf{E}_{ext}(\mathbf{r}_c, t), \quad (2.12)$$

where $\mathbf{r}_c$ is the charge centre, and $\mathbf{g}_c$ its acceleration. Moreover, q is the total charge, and M the total mass of the system. Thus, the result (2.12) represents a precise basic equation of motion of a charged system. It gives an essential modification of the so-called Abraham-Lorentz equation; the standard factor 4/3 multiplying the electromagnetic mass has disappeared, because equivalence reigns in M. In addition, eq. (2.12) contains the subtlety that the system is represented by one definite point, i.e. the charge centre $\mathbf{r}_c$. Consequently, eq. (2.12) can be used as an equation of motion for, say, a uranium nucleus in an external electric

field, or, for a classical electron model. One may immediately correct the equation by a familiar term containing radiation damping, if so desired.

Although, as mentioned, eq. (2.12) is widely applicable, let us register main corrections to it, or assumptions contained in it. We have assumed that the external field varies slowly in time. The time variation of $\mathbf{E}_{ext}$ will, first, give rise to adiabatic changes of M and, second, to non-adiabatic mass excitations. Third, we have already mentioned that a time variation of $\mathbf{g}_c$ leads to radiation damping. Finally, if $\mathbf{g}_c$ quickly changes direction, the mass centre need not remain behind the charge centre, and rotations may be induced. In order to study that, systems with spin must be included in the description, but that is outside the scope of the present paper.

Shortcomings of standard formulae

At this stage it is convenient to compare with the standard derivation of self-force as alluded to in §1, and performed by Heitler, for instance. In that description, the self-fields are again given by (2.8), but with $\mathbf{g}_k = \mathbf{g}_0$, which does not lead to immediate errors. Next, the total force is calculated, corresponding to (1.8) and (2.5), and erroneously identified with the mass times the acceleration $\mathbf{g}_0$. Thus, from (2.8) and (1.8)

$$\mathbf{F} = \sum_i \mathbf{F}_i = \sum_i m_i \mathbf{g}_0 + \sum_{i,k}^{i \neq k} \frac{1}{2 r_{ik} c^2} \left(\mathbf{g}_0 + \frac{(\mathbf{g}_0 \mathbf{r}_{ik})}{r_{ik}^2} \mathbf{r}_{ik} \right), \tag{2.13}$$

where the dominant Coulomb terms have cancelled out, in contrast to (2.9).

For a spherical symmetric charge distribution, with electrostatic energy U, the expression (2.13) leads to

$$\mathbf{F} = \left(\sum_i m_i + \frac{4}{3} \frac{U}{c^2} \right) \mathbf{g}_0 , \tag{2.14}$$

In case the charge distribution is merely symmetric about the direction of $\mathbf{g}_0$, the factor 4/3 is seen to be replaced by a number $1 + \xi$, where $0 \leq \xi \leq 1$. For a general distribution, however, the force $\mathbf{F}$ need not even point in the direction of $\mathbf{g}_0$. We have hereby clarified in some detail the shortcomings of the Galilean concept of a total force and its association with total mass, in special relativity.

Next, let us consider the standard formulae for self-momentum and self-energy of a charged system, i.e. (1.8) and (1.9). Although they are connected to the above-mentioned standard self-force calculation, their short-comings are of a more elusive kind. Still, in order to elucidate their basic contents, we can observe the following. If a system has internal Coulomb energy U, then the standard self-momentum corresponding to (1.8) can be $\mathbf{P}_{el} = (U/c^2) \mathbf{v} \cdot \gamma \cdot (1 + \xi)$, where

$\xi = 1/3$ for a spherically symmetric charge distribution. Similarly, the standard self-energy becomes $E_{el} = U \cdot \gamma \cdot (1 + \xi v^2/c^2)$, corresponding to (1.9). Therefore, if we form the difference $\mathbf{v} \cdot \mathbf{P}_{el} - E_{el}$, we invariably get a quantity independent of ξ, namely

$$L_{el} = -E_{el} + \mathbf{v} \cdot \mathbf{P}_{el} = -U \cdot \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{1/2}. \tag{2.15}$$

In point of fact, we have recovered eq. (1.4), i.e. the Lagrangian belonging to equivalence. Moreover, it is apparent that, instead of making the indirect derivation of (2.15) via E_{el} and $\mathbf{P}_{el}$, we could have obtained it directly by integration of the invariant Lagrangian density in space, as belonging to the field and to its interaction with matter, or $\mathscr{L}_{field} + \mathscr{L}_{int}$.

Returning to the Lagrangian in (2.15), we have already seen, in (1.5), that if we vary L_{el} with respect to $\mathbf{v}$, keeping the internal state unchanged, we arrive at the momentum $\mathbf{P} = (U/c^2)\gamma \cdot \mathbf{v}$, leading to equivalence. In fact, this implies *detailed* equivalence, the Coulomb contribution being only part of the total internal energy.

Next, it also becomes clear how the standard momentum $\mathbf{P}_{el}$ can be connected to (2.15): when $\mathbf{v}$ is varied, there is assumed to be an associated variation of the internal state – i.e. of U. Thus, $\mathbf{P}_{el}$ will result if we let U vary proportionally to $(1 - v^2/c^2)^{\xi/2}$, when L_{el} is varied with respect to $\mathbf{v}$. The factor in question must be due to a Lorentz transformation of the internal variables of the system.

We have thus realized that the standard momentum and energy (1.8) and (1.9), arise from an unwarranted variational procedure, whereby they lose connection to our basic concepts of momentum and energy of an isolated system, as described in §1. Such results arise in general from arbitrary transformations of internal variables of a system, where momenta become abstract quantities, without direct physical significance.

Self-mass for point force

The present discussion of the central ingredients in equivalence calculations is perhaps best concluded by means of an example serving a triple purpose. First, it concerns an external point force, implying the simplest possible connection to mass and acceleration. Second, the equivalence in question applies to deformation energies, not studied explicitly above. Third, Poincaré stresses cannot be introduced.

Let two mutually repelling charges, q_a and q_b, be accelerated from rest along their line of connection, $\mathbf{r}_{ba}$. An external force $\mathbf{F}_a$ acts on particle a, while particle b in turn is made to accelerate by the Coulomb repulsion from particle a. The

force and distance r_{ab} are balanced such that the particles are accelerated in rigid motion. The internal energy of the system, $q_a q_b / r_{ab}$, is purely an energy of deformation. The internal electric fields are given by (2.8), and so the equations of motion of the two particles become

$$m_a \mathbf{g}_a = \frac{q_a q_b}{r_{ab}^3} \mathbf{r}_{ab} - \frac{q_a q_b}{r_{ab} c^2} \mathbf{g}_b + \mathbf{F}_a \,, \tag{2.16}$$

$$m_b \mathbf{g}_b = \frac{q_a q_b}{r_{ab}^3} \mathbf{r}_{ba} - \frac{q_a q_b}{r_{ab} c^2} \mathbf{g}_a \,, \tag{2.17}$$

where all vectors are collinear, while m_a and m_b are mechanical masses. Further, the condition of rigid acceleration determines $\mathbf{g}_b$ in terms of $\mathbf{g}_a$ and $\mathbf{r}_{ba}$, cf. (2.3),

$$\mathbf{g}_b = \frac{\mathbf{g}_a}{1 + \mathbf{g}_a \cdot \mathbf{r}_{ba} / c^2} \,. \tag{2.18}$$

We multiply (2.17) by $1 + (\mathbf{g}_a \cdot \mathbf{r}_{ba})/c^2$, add (2.16), and obtain to first order in $\mathbf{g}_a$

$$\mathbf{F} = \mathbf{g}_a \left(m_a + m_b + \frac{q_a q_b}{r_{ab}} \frac{1}{c^2} \right). \tag{2.19}$$

Since the mass M of the system must be given by $M \mathbf{g}_a = \mathbf{F}_a$, we find

$$M = m_a + m_b + \frac{q_a q_b}{r_{ab}} \frac{1}{c^2} \,. \tag{2.20}$$

In the simplest imaginable case we have thus obtained equivalence, and for a deformation energy in fact.

Let us next turn to the standard procedure, where the total self-force is calculated, cf. (2.13) or (1.10). It corresponds to adding the right-hand sides of (2.16) and (2.17), omitting $\mathbf{F}_a$. The self-mass becomes erroneous, or $2q_a q_b/(r_{ab} c^2)$, like in (2.13). We might similarly, as done by Heitler, add (2.16) and (2.17) with the Galilean demand $\mathbf{g}_b = \mathbf{g}_a$, and obtain a wrong value of the force $\mathbf{F}_a$. In any case, there are here no compensating Poincaré stresses, which can repair the error.

Whereas, in the present chapter, we have arrived at the proper treatment belonging to coordinates (inertial frames) where equations of motion are simple but self-mass calculations delicate, the theme in the following will be reversed. In fact, we shall introduce coordinates (Møller box) where self-mass calculations become extremely simple; our task will be to obtain the equations of motion. Thereby, the discussion becomes lengthy, containing transformations of equations of motion in various classical and quantal cases.

§ 3. Systematic Description by Means of Accelerated Reference Frame

Basic properties of Møller box

The previous chapter contained a preliminary analysis of simple acceleration processes for which the intrinsic state remained stationary in the instantaneous rest frame. A systematic analysis of such processes must be based on a description of the system in an accelerated rigid frame of reference, always coinciding with the instantaneous rest frame. A coordinate system of this kind, where each point has a time independent acceleration in its momentary rest frame, we shall refer to as a Møller box *).

Before embarking on a detailed discussion of the Møller box, we may point out some of its salient features. First, the physical laws in the box are independent of time, i.e. there is invariance against time displacement and time-reflection. Second, there is an inborn simultaneity, like in a static gravitational field. Third, it also follows that a charge at rest in the Møller box gives rise to a purely electrostatic field in this frame. This feature corresponds to the fact that for a static charge in the box, performing a hyperbolic motion in an inertial frame, the retarded and advanced fields are identical within the box. Fourth, in the inertial frame we had to distinguish between, on the one hand, that weighted sum of forces which leads to the total mass of a system and, on the other hand, the total force. In the Møller box these concepts are united in the sense that the total force required to keep a body at rest in the box is proportional to the total mass of the body.

When introducing the Møller box it is useful to consider first the hyperbolic motion of a single particle. Let a particle in an inertial frame K be accelerated along the X-axis, with the constant acceleration g_0 in its rest frame. It is convenient to introduce the length $\lambda = c^2/g_0$ and write for the trajectory

$$X_0 = \lambda \cosh \frac{ct}{\lambda}, \quad Y_0 = Z_0 = 0, \tag{3.1}$$

$$cT_0 = \lambda \sinh \frac{ct}{\lambda}.$$

Thus, the coordinates cT_0 and X_0 lie on the hyperbola

$$X_0^2 - c^2 T_0^2 = \lambda^2. \tag{3.2}$$

*) Accelerated rigid frames of reference are discussed in some detail by C. Møller in his monograph on relativity[13]. We have adopted the name Møller box for the particular frame discussed in the text. The wording *box* is meant to indicate that we are considering a space-time domain of finite extension.

Here (T_0, X_0, Y_0, Z_0) are the particle coordinates in the inertial frame K, and t denotes proper time for the particle. For this motion the four-velocity

$$U_0 = \left(\frac{d(cT_0)}{dt}, \frac{dX_0}{dt}, 0,0\right) = \frac{1}{\lambda}(X_0, cT_0, 0,0) \tag{3.3}$$

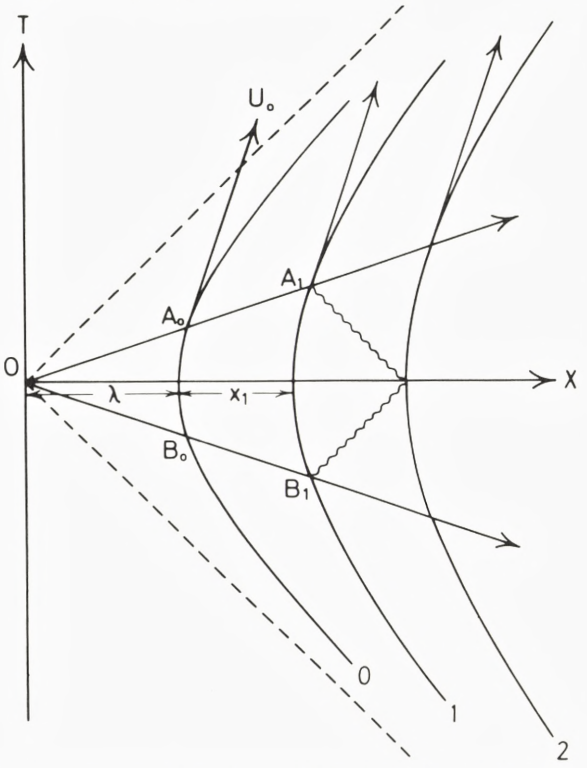

Fig. 2. The world lines of the reference points 0,1 and 2 in the Møller box are scaled hyperbolae with the light cone through the origin as the common asymptote. Successively steeper lines through the origin correspond to progressive time in the Møller box. Proper time on the hyperbola labelled 0 has been chosen as the common standard time in the box, and distances are measured from this hyperbola. The wavy lines represent the world line of a light signal exchanged between the observers 1 and 2.

is always perpendicular to the radius vector from the origin $(cT_0, X_0, 0,0)$. It is therefore clear from Figure 2 that if another particle is to remain at rest relative to the first at the distance x_1, its world line will be given by

$$(cT_1, X_1, 0,0) = \left(1+\frac{x_1}{\lambda}\right)(cT_0, X_0, 0,0) \tag{3.4}$$

i.e. it will also perform a hyperbolic motion, but with rest acceleration $g_0(1+g_0x_1/c^2)^{-1}$. Thus we are led to the introduction of the accelerated rigid frame, the Møller box, with coordinates (t,x,y,z), by the transformation from the variables (T,X,Y,Z) in the inertial frame K:

$$\begin{aligned} X &= (x+\lambda)\cosh\frac{ct}{\lambda}, \quad Y = y, \quad Z = z, \\ cT &= (x+\lambda)\sinh\frac{ct}{\lambda}, \end{aligned} \tag{3.5}$$

corresponding to the line element

$$ds^2 = c^2 dT^2 - d\mathbf{R}^2 = \left(1 + \frac{x}{\lambda}\right)^2 \cdot c^2 dt^2 - d\mathbf{r}^2 . \tag{3.6}$$

For $T = t = 0$ we have $X = x + \lambda$, where we have selected a standard hyperbola from which we measure distances along the x-direction in the box. The proper time t of the standard hyperbola $x = 0$ has been chosen as the common standard time variable in the box. Hence, outside the standard hyperbola, local proper time deviates from standard time. In fact, the relation between standard time t and local proper τ at the position x is given by

$$d\tau = \left(1 + \frac{x}{\lambda}\right) dt . \tag{3.7}$$

The choice of the standard time t in the Møller box makes explicit the invariance against time displacement and time reflection inherent in this static reference frame. Therefore, this way of synchronizing events corresponds to the inborn simultaneity in the box. This can be illustrated by considering a light signal moving along the x-axis between two observers 1 and 2 at rest in the box (see figure 2). We assume that the light signal is sent back at time $T = 0$. We notice that since each hyperbola corresponds to the locus of constant distance from O in Minkowski space, they are symmetric with respect to the radius vectors in this space. We have therefore drawn the figure such that the inherent symmetry of the hyperbolae is made explicit with respect to the axis $T = 0$. Events on the line OB_0B_1 are simultaneous with the departure of the light signal from the first observer, and events on the line OA_0A_1 are simultaneous with its return. It is obvious from the figure that if the events are synchronized to proper time of the standard hyperbola $x = 0$, the time of arrival to the second observer will be halfway between the time of departure from the first observer and the time of return. This result is also directly borne out by evaluation of the standard time intervals in question, which are found to be $(\lambda/c)\log[(x_2 + \lambda)/(x_1 + \lambda)]$, x_1 and x_2 being the coordinates of the two observers.

It follows from the transformation (3.5) that a particle at rest in the Møller box at position x at time $T = t = 0$ has acceleration

$$g(x) = \frac{c^2}{\lambda + x} = \frac{g_0}{1 + g_0 x/c^2} \tag{3.8}$$

in the inertial frame. This is of course the result (2.3) already deduced for the rigid acceleration in the inertial frame. In contrast, a particle at rest at $T = t = 0$ in the inertial frame at the same position $X = x + \lambda$, in the Møller box has an acceleration

$$\frac{d^2x}{dt^2} = -g_0\left(1+\frac{g_0 x}{c^2}\right). \tag{3.9}$$

Thus (3.9) expresses the acceleration of a freely falling particle in the coordinates of the Møller box. It has of course the opposite sign of (3.8), but, more important, its magnitude increases in the direction of g_0 in contrast to $g(x)$.

The line element (3.6) implies a simple scaling law for velocities. In particular, the velocity of light in the box is given by

$$c(x) = c\cdot\left(1+\frac{g_0 x}{c^2}\right). \tag{3.10}$$

In fact, whereas the velocity of light in local units is alsays c, it must be changed by the factor in (3.7) when we measure in standard time.

Electrostatic interactions in Møller Box

In this section we shall be concerned with charged particles moving in static potentials*). The field equations for static potentials in the Møller box are derived from the action principle

$$\delta S_f + \delta S_{int} = 0\,, \tag{3.11}$$

where the contributions to the action from the field and the interaction are obtained from the general expressions (A13) and (A14):

$$S_f = -\frac{1}{8\pi}\int dt\int d^3r\,\frac{(\nabla\varphi)^2}{(1+x/\lambda)}\,, \tag{3.12}$$

$$S_{int} = -\int dt\int d^3r\,\varrho(\mathbf{r})\,\varphi(\mathbf{r})\,. \tag{3.13}$$

If (3.11) corresponds to variation of the potential φ for fixed charge distribution ϱ, one obtains to first order in $\lambda^{-1} \equiv g_0/c^2$ the result

$$\Delta\varphi(\mathbf{r}) - \frac{1}{\lambda}\frac{\partial\varphi(\mathbf{r})}{\partial x} = -4\pi\left(1+\frac{x}{\lambda}\right)\varrho(\mathbf{r}). \tag{3.14}$$

*) For the sake of completeness, a discussion of electrodynamics and the equations of motion of charged particles in the Møller box is given in appendix A, while the static potentials are solved in appendix B.

Consider the potential $\varphi(\mathbf{r};\mathbf{r}_1)$ in the point $\mathbf{r}$ generated by the charge density $\varrho(\mathbf{r}) = q_1\delta(\mathbf{r}-\mathbf{r}_1)$, i.e. by a single charge q_1 at rest in the point $\mathbf{r}_1$. According to (3.14) this potential is to first order in $1/\lambda$

$$\varphi(\mathbf{r};\mathbf{r}_1) = \frac{q_1}{|\mathbf{r}-\mathbf{r}_1|}\left(1+\frac{x+x_1}{2\lambda}\right). \tag{3.15}$$

In order to appreciate the significance of the second term in the brackets in (3.15), consider a second charge q_2 at rest at the position $\mathbf{r}_2$. We notice that the density of the interaction energy of two charges depends on the values of the two charges only through their product. We therefore expect that the position of the centre-of-mass of the interaction energy is always the same as for two identical charges, namely midway between them. Thus, in the interaction energy $q_2\,\varphi(\mathbf{r}_2;\mathbf{r}_1)$ we can interpret the term

$$\delta E_{int} = \frac{q_1 q_2}{|\mathbf{r}_2-\mathbf{r}_1|c^2} g_0 \frac{x_2+x_1}{2} \tag{3.16}$$

as the potential energy of a mass $q_1q_2/c^2|\mathbf{r}_2-\mathbf{r}_2|$ located at the midpoint between the charges in the artificial gravitational field g_0. The energy (3.16) therefore represents the work required to lift a mass equivalent to the Coulomb energy from the reference level $x = 0$ to the height $(x_1+x_2)/2$. Hence we expect that the sum of the mutual forces be equal to $(q_1q_2/c^2|\mathbf{r}_2-\mathbf{r}_1|)g_0$. This is indeed in accordance with (3.15), from which it follows that

$$q_2\frac{\partial\varphi(\mathbf{r}_2;\mathbf{r}_1)}{\partial x_2} + q_1\frac{\partial\varphi(\mathbf{r}_1;\mathbf{r}_2)}{\partial x_1} = \frac{q_1q_2}{|\mathbf{r}_2-\mathbf{r}_1|c^2} g_0 \,. \tag{3.17}$$

This relation, which embodies the equivalence between electrostatic energy and mass, will be crucial in the following discussion. It corresponds to the weighted addition of forces (2.6), applied to the Coulomb case (2.8), but in the Møller box it emerges as a direct consequence of the term (3.16) in the interaction energy.

The equations of motion in the Møller box, for a particle of mass m and charge q moving in a static potential φ, is obtained from the action principle

$$\delta S_{kin} + \delta S_{int} = 0, \tag{3.18}$$

with

$$S_{kin} = -mc^2\int d\tau = -mc^2\int dt\left\{\left(1+\frac{x}{\lambda}\right)^2 - \frac{\mathbf{v}^2}{c^2}\right\}^{1/2}, \tag{3.19}$$

$$S_{int} = -\int dt\, q\varphi(\mathbf{r}(t)) \,. \tag{3.20}$$

If (3.18) corresponds to variation of the particle coordinates for fixed potential, one obtains the equations of motion of the particle. They correspond to the Lagrangian

$$L = -mc^2 \left\{ \left(1 + \frac{x}{\lambda}\right)^2 - \frac{\mathbf{v}^2}{c^2} \right\}^{1/2} - q\varphi(\mathbf{r}) \,. \tag{3.21}$$

Equivalence for kinetic energy

The first term in the Lagrangian (3.21) turns out to contain separate equivalence for kinetic energy. In order to illustrate this property let us consider a ball of mass m, bouncing between ceiling and floor of a small rectangular enclosure which is kept at rest in the Møller box, with its floor at the reference level $x = 0$, and with its edges parallel to the coordinate axis. Let the ball jump from the floor with momentum $p_x(1)$ in the x-direction and hit the ceiling after a time T with momentum $p_x(2)$ along this direction. We assume that the enclosure is so small that the kinetic energy of the ball can be regarded as constant during its motion. Consequently the net momentum transfer per unit time from the ball to the box is

$$\frac{p_x(2) - p_x(1)}{T} = \frac{1}{T}\int_0^T dt \frac{dp_x}{dt} = \frac{1}{T}\int_0^T dt \frac{\partial L}{\partial x} = -\frac{mg_0}{(1 - \mathbf{v}^2/c^2)^{1/2}} \,, \tag{3.22}$$

where we have used the equation of motion corresponding to the first term of the Lagrangian (3.21). In order to keep the enclosure at rest in the Møller box, the presence of the bouncing ball thus requires an extra force $\delta\mathcal{F}$ so as to support the floor of the enclosure

$$\delta\mathcal{F} = \frac{m}{(1 - \mathbf{v}^2/c^2)^{1/2}} \mathbf{g}_0 \,. \tag{3.23}$$

This force is the same in the Møller box as in the particular inertial frame, which momentarily coincides with the enclosure, since standard time coincides with proper time at the location $x = 0$.

However straightforward this demonstration of separate equivalence for kinetic energy may appear, one should note that it stands in contrast to the conventional treatment of similar examples. In fact, in the latter approach equivalence can only be stablished by explicitly taking into account the stresses set up in the walls by the bouncing ball, and would not apply to the kinetic energy separately.

§ 4. Equivalence for Atomic Binding Energies

So far we have discussed quite idealized systems in which there was either electrostatic or kinetic energy present. As a simple example of a more realistic physical system with both kinetic and potential energy, we consider a hydrogen atom and enquire into the total force necessary to keep the nucleus at rest at the position $\mathbf{r}_n = 0$ in the Møller box. This force is the same in the Møller box as in that inertial frame in which the nucleus is momentarily at rest, since standard time coincides with proper time at the position $\mathbf{r} = 0$. For simplicity we treat the atom as a one-particle system, i.e. we neglect the motion of the heavy nucleus around the mass centre.

The atom is assumed to be in a stationary state in the Møller box. Within quantum mechanics, this means that the wave function corresponds to a definite energy in the Møller box and the associated charge distribution of the electron is static in this frame. In the case of classical mechanics we are dealing with definite orbits, time-averages over which correspond to quantum mechanical expectation values. One would expect that the question of equivalence be independent of whether a quantal or a classical description is used in accounting for the stability of the system. This is indeed borne out by the following discussion.

Classical hydrogen atom

The external force, $\mathcal{F}$, required to keep the nucleus with charge Ze and mass m_n at rest in the box, must compensate the fictitious gravitational force $-m_n \mathbf{g}_0$ acting on the nucleus in the accelerated frame, as well as the reaction force on the nucleus from the electron of charge $-e$ and mass m_e.

According to the Lagrangian (3.21) the time average, F_x, of this reaction force over a time T, long compared to the orbital periods of the atom, is

$$F_x = \frac{1}{T}\int_0^T dt\, Ze \left.\frac{\partial \varphi(\mathbf{r}_n;\mathbf{r}_e(t))}{\partial x_n}\right|_{\mathbf{r}_n=0}, \tag{4.1}$$

where $\mathbf{r}_e(t)$ is the coordinate of the electron. From the relation (3.17) and the Lagrangian (3.21) we find for the integrand in (4.1)

$$\begin{aligned} Ze\frac{\partial \varphi(\mathbf{r}_n;\mathbf{r}_e)}{\partial x_n} &= \frac{Ze(-e)}{\lambda|\mathbf{r}_e-\mathbf{r}_n|} - (-e)\frac{\partial \varphi(\mathbf{r}_e;\mathbf{r}_n)}{\partial x_e} \\ &= -\frac{Ze^2}{\lambda|\mathbf{r}_e-\mathbf{r}_n|} + \frac{\partial L}{\partial x_e} + \frac{\partial}{\partial x_e} m_e c^2 \left\{\left(1+\frac{x_e}{\lambda}\right)^2 - \frac{\mathbf{v}_e^2}{c^2}\right\}^{1/2}. \end{aligned} \tag{4.2}$$

Because of the equations of motion, the term

$$\frac{\partial L}{\partial x_e} = \frac{d}{dt}\frac{\partial L}{\partial v_{ex}}$$

does not contribute to the time average (4.1). Hence, keeping only terms of order $1/\lambda$, we obtain an average reaction force

$$F_x = -\frac{1}{T}\int_0^T dt\left[\frac{m_e c^2}{(1-\mathbf{v}_e^2/c^2)^{1/2}} - \frac{Ze^2}{|\mathbf{r}_e - \mathbf{r}_n|}\right]\frac{g_0}{c^2}. \tag{4.3}$$

We notice that the integrand in (4.3) is the energy of the electron multiplied by g_0/c^2, and accordingly time independent. Thus the total force required to keep the atom at rest is

$$\mathcal{F} = \left[m_n + \frac{m_e}{(1-\mathbf{v}_e^2/c^2)^{1/2}} - \frac{Ze^2}{|\mathbf{r}_e - \mathbf{r}_n|c^2}\right]\mathbf{g}_0\,, \tag{4.4}$$

or

$$\mathcal{F} = \left(m_n + m_e - \frac{B}{c^2}\right)\mathbf{g}_0\,, \tag{4.5}$$

where B is the binding energy of the atom. The relation (4.5) expresses the equivalence between binding energy and mass for a classical hydrogen atom to all orders in v/c. The limitations to this result are solely due to the possible radiation from the system, proportional to some power of e^2. The classical orbits depend on the charges through their product only. We can therefore consider (4.5) as an exact result for a given orbital configuration of the atom, corresponding to the limit $e^2 \to 0$ for fixed value of the product Ze^2, in which limit the radiation is negligible.

Quantal hydrogen atom

The above discussion of a classical hydrogen atom can be carried over to the quantal case by passing from a Langrangian to a Hamiltonian description and replacing time averages by expectation values. Whereas the motion of the nucleus is still treated in classical terms, the state of the electron is now described by the Hamiltonian operator constructed from the Lagrangian (3.21):

$$\begin{aligned} H &= \frac{1}{2}\left\{\left(1+\frac{x_e}{\lambda}\right)c[m_e^2c^2+\mathbf{p}_e^2]^{1/2} + c[m_e^2c^2+\mathbf{p}_e^2]^{1/2}\left(1+\frac{x_e}{\lambda}\right)\right\} - e\varphi(\mathbf{r}_e;\mathbf{r}_n) \\ &\equiv H_0 - e\varphi(\mathbf{r}_e;\mathbf{r}_n)\,. \end{aligned} \tag{4.6}$$

Here the operators $\mathbf{r}_e$ and $\mathbf{p}_e$ refer to the electron and satisfy the usual commutation relation, whereas $\mathbf{r}_n$, the coordinate of the nucleus, is a c-number. The first term, H_0, has been symmetrized in an obvious manner, and the potential $\varphi(\mathbf{r}_e;\mathbf{r}_n)$

generated by the nucleus, is given by (3.15). We use here the Hamiltonian (4.6) in order to emphasize the analogy to the classical treatment. As indicated below, similar considerations can be applied to the Dirac Hamiltonian, whereby effects associated with the electron spin are included.

In close analogy to (4.2) we obtain from (4.6)

$$Ze\frac{\partial\varphi(\mathbf{r}_n;\mathbf{r}_e)}{\partial x_n} = -\frac{Ze^2}{\lambda|\mathbf{r}_e-\mathbf{r}_n|} + e\frac{\partial\varphi(\mathbf{r}_e;\mathbf{r}_n)}{\partial x_e} = -\frac{Ze^2}{\lambda|\mathbf{r}_e-\mathbf{r}_n|} - \left[\frac{\partial}{\partial x_e}, H-H_0\right]. \quad (4.7)$$

Since the expectation value of $\left[\frac{\partial}{\partial x_e}, H\right]$ in the last term of (4.7) vanishes in a stationary state, we get for the expectation value, F_x, of the reaction force

$$F_x = -<\psi\left|Ze\frac{\partial\varphi(\mathbf{r}_n;\mathbf{r}_e)}{\partial x_n}\right|\psi> = -<\psi\left|c[m_e^2c^2+\mathbf{p}_e^2]^{1/2} - \frac{Ze^2}{|\mathbf{r}_e-\mathbf{r}_n|}\right|\psi> \frac{g_0}{c^2}$$

$$= -\left(m_e - \frac{B}{c^2}\right)g_0 . \quad (4.8)$$

Thus, the total force required to keep the atom at rest is given by eq. (4.5).

The Dirac equation

In order to establish the form of the Dirac equation in the Møller box we notice that for any value of the standard time t, the wave function $\psi(t)$ in the Møller box is equal to the wave function $\psi_{K(t)}(T)$ in that inertial frame $K(t)$ which at time t coincides with the box

$$\psi(t) = \psi_{K(t)}(T) . \quad (4.9)$$

Here T is the time measured in the inertial frame $K(t)$ and, according to (3.7), the time intervals dt and dT are related by

$$dT = \left(1+\frac{x}{\lambda}\right)dt . \quad (4.10)$$

The time variation of $\psi(t)$ is due partly to the change of inertial frame $K(t)$ with time and partly to the intrinsic time variation of the state. In order to find the change with time, t, of the wave function at a fixed space point in the box, let us consider the three events pictured in Fig. 3. Here the events 1 and 2 refer to one and the same space point in the box, but are separated by the time interval δt in this frame. Similarly 2 and 3 are simultaneous in the box, whereas 1 and 3 refer to one and the same space point in the inertial frame $K(t)$, but are separated by the time interval δT in this frame.

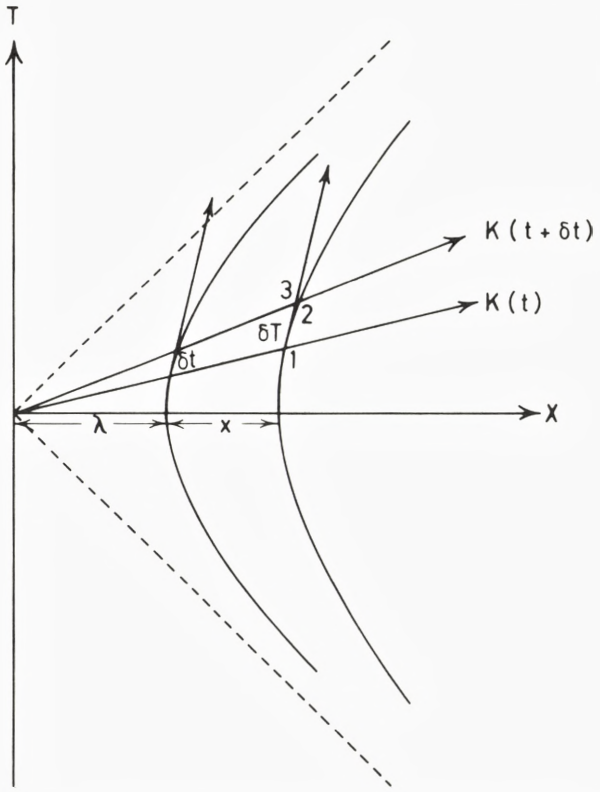

Fig. 3. Space-time diagram showing the events 1, 2 and 3, mentioned in the text, and two successive rest frames $K(t)$ and $K(t+\delta t)$. The events 1 and 2 refer to one and the same space point in the Møller box, whereas 1 and 3 refer to one and the same space point in the the inertial frame $K(t)$.

For the sake of simplicity we use as variables in the wave function the numbers referring to the appropriate space-time points. With this notation we get

$$\psi(2)-\psi(1)\equiv\psi_{K(t+\delta t)}(2)-\psi_{K(t)}(1)=\left(\psi_{K(t+\delta t)}(2)-\psi_{K(t)}(3)\right)+\left(\psi_{K(t)}(3)-\psi_{K(t)}(1)\right) \tag{4.11}$$

Since the separation between the points 2 and 3 is of the order $g_0(\delta t)^2$, we get to first order in δt

$$\psi(2)-\psi(1)=\left(\psi_{K(t+\delta t)}(2)-\psi_{K(t)}(2)\right)+\frac{\partial\psi_{K(t)}}{\partial T}\delta T. \tag{4.12}$$

Because the inertial frame $K(t+\delta t)$ moves with the infinitesimal velocity $\mathbf{v}=\mathbf{g}(x)\delta T=\mathbf{g}_0\delta t$ relative to the inertial frame $K(t)$, we obtain

$$\psi_{K(t+\delta t)}(2)\simeq\exp\left(\frac{-\boldsymbol{\alpha}\mathbf{v}}{2c}\right)\psi_{K(t)}(2)\simeq\left(1-\frac{\boldsymbol{\alpha}\mathbf{g}}{2c}\delta t\right)\psi_{K(t)}(2) \tag{4.13}$$

Here the matrix $\exp(-\frac{1}{2}\boldsymbol{\alpha}\mathbf{v}/c)$, where $\boldsymbol{\alpha}$ denotes the usual Dirac matrices,

transforms the wave function from the inertial frame K(t) to K(t+δt). Combining (4.10), (4.12) and (4.13), we obtain

$$i\hbar \frac{\partial \psi}{\partial t} = -i\hbar \frac{\boldsymbol{\alpha} \mathbf{g}_0}{2c} \psi + \left(1 + \frac{x}{\lambda}\right) i\hbar \frac{\partial \psi}{\partial T} = -i\hbar \frac{\boldsymbol{\alpha} \mathbf{g}_0}{2c} \psi + \left(1 + \frac{x}{\lambda}\right) H_D \psi . \tag{4.14}$$

Here H_D is the Dirac Hamiltonian in the inertial frame K(t),

$$H_D = c\boldsymbol{\alpha} \left(\mathbf{p} + \frac{e}{c} \mathbf{A}_{K(t)}\right) + \beta mc^2 - e\varphi_{K(t)} , \tag{4.15}$$

where $(\varphi_{K(t)}, \mathbf{A}_{K(t)})$ is the four-potential in the frame K(t), and where the electron charge is $-e$. The Møller box coincides with K(t) at time t, and therefore

$$\mathbf{p} = -i\hbar \nabla_{\mathbf{r}} , \tag{4.16}$$

where $\mathbf{r}$ denotes the spatial coordinates in the box. From the equations (4.14)-(4.16) it follows that the Dirac equation in the Møller box takes the form, valid to all orders in $1/\lambda$,

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2} \left[\left(1 + \frac{x}{\lambda}\right) H_D + H_D \left(1 + \frac{x}{\lambda}\right) \right] \psi . \tag{4.17}$$

From eq. (4.17) one may derive the continuity equation

$$\frac{\partial}{\partial t} (\psi^+ \psi) + \operatorname{div}_{\mathbf{r}} \left(\psi^+ \left(1 + \frac{x}{\lambda}\right) c\boldsymbol{\alpha} \psi \right) = 0 . \tag{4.18}$$

Therefore, the quantity $(-e\psi^+\psi)$ is the charge density in the Møller box and $(-e\psi^+(1+x/\lambda)\boldsymbol{\alpha} c\psi)$ is the charge current density.

In order to apply the Dirac equation to a hydrogen atom with the nucleus at rest at $\mathbf{r} = 0$, we have to find the potentials $\varphi_{K(t)}$ and $\mathbf{A}_{K(t)}$ generated by the nucleus. Because the inertial frame K(t) is the momentary rest frame of the nucleus at time t, the retarded potentials are to first order in the acceleration given by (cf. ref. 12, p. 167)

$$\begin{aligned} \varphi_{K(t)} &= \frac{Ze}{r} + \frac{Ze}{2c^2} \frac{d^2 r}{dT^2} , \\ \mathbf{A}_{K(t)} &= 0 , \end{aligned} \tag{4.19}$$

where $r = |\mathbf{r}_e - \mathbf{r}_n|$ and d^2r/dT^2 refers to the inertial frame K(t), i.e.

$$\frac{d^2 r}{dT^2} = -\frac{\mathbf{g}_0 \cdot (\mathbf{r}_e - \mathbf{r}_n)}{r} . \tag{4.20}$$

Thus we obtain

$$\varphi_{K(t)} = \frac{Ze}{|\mathbf{r}_e - \mathbf{r}_n|}\left(1 - \frac{\mathbf{g}_0 \cdot (\mathbf{r}_e - \mathbf{r}_n)}{2c^2}\right),$$
$$\mathbf{A}_{K(t)} = 0\,. \tag{4.21}$$

From (4.15) and (4.17) we then get

$$i\hbar \frac{\partial \psi}{\partial t} = \left\{\frac{1}{2}\left(1 + \frac{\chi_e}{\lambda}\right)(c\boldsymbol{\alpha} \cdot \mathbf{p}_e + \beta m_e c^2) + \frac{1}{2}(c\boldsymbol{\alpha} \cdot \mathbf{p}_e + \beta m_e c^2)\left(1 + \frac{\chi_e}{\lambda}\right) - e\varphi(\mathbf{r}_e; \mathbf{r}_n)\right\}\psi \tag{4.22}$$

Here the potential

$$\varphi(\mathbf{r}_e; \mathbf{r}_n) = \left(1 + \frac{\chi_e}{\lambda}\right)\varphi_{K(t)}(\mathbf{r}_e; \mathbf{r}_n) \tag{4.23}$$

is seen to be identical with the electrostatic potential in the Møller box as given, to first order in $1/\lambda$, by (3.15). The Hamilton in (4.22) is just what one would obtain by simply replacing the square root in the Hamiltonian (4.6) by $(c\boldsymbol{\alpha} \cdot \mathbf{p}_e + \beta m_e c^2)$.

Since $(-e\psi^+\psi)$ is the charge density of the electron in the atom, we can immediately write down the expectation value, F_x, of the reaction force on the nucleus from the electron in a stationary state. By steps analogous to those of eq. (4.7), we obtain

$$\begin{aligned} F_x &= -\langle \psi \left| Ze \frac{\partial \varphi(\mathbf{r}_n; \mathbf{r}_e)}{\partial x_n}\right|_{\mathbf{r}_n = 0} | \psi \rangle = \\ &= -\langle \psi \left| \frac{Ze^2}{\lambda |\mathbf{r}_e - \mathbf{r}_n|}\right| \psi \rangle + \langle \psi \left| e \frac{\partial \varphi(\mathbf{r}_e; \mathbf{r}_n)}{\partial x_e}\right| \psi \rangle \\ &= -\langle \psi \left| c\boldsymbol{\alpha} \cdot \mathbf{p}_e + \beta m_e c^2 - \frac{Ze^2}{|\mathbf{r}_e - \mathbf{r}_n|}\right| \psi \rangle \frac{g_0}{c^2} \\ &= -\left(m_e - \frac{B}{c^2}\, g_0\right). \end{aligned} \tag{4.24}$$

Thus the total force required to keep the atom at rest is given by the expression (4.5), where B, the binding energy of the electron, now includes spin-orbit coupling, the Darwin term and all other effects contained in the Dirac Hamiltonian.

It is also possible to demonstrate equivalence for a hydrogen-like atom described by the Klein-Gordon equation. Since the argumentation is somewhat different from the cases considered so far, the Klein-Gordon equation is treated separately in appendix C.

Thomas-Fermi atom

In the previous analysis we have considered equivalence for electrostatic energies and kinetic energies of a single particle bound in an atom. When we turn to more general systems, consisting of several moving charged particles, one might attempt to base the discussion on a mechanical description in terms of the coordinates and velocities of the particles only. It has turned out, however, that this description can in general only be carried to terms proportional to $1/c^2$, within an expansion in powers of $1/c$. The corresponding Lagrangian is the familiar one introduced by Darwin (cf. Landau and Lifshitz[12]). Equivalence may be demonstrated within this scheme, but a strong limitation is then imposed on the internal velocities of the system as well as on the velocity belonging to Lorentz transformations. Such limitations are avoided in a self-consistent description of the system, in which each particle interacts with a common four-potential, the latter being generated by the particles themselves. As a first step towards such general dynamical descriptions we shall study equivalence for the simple case of a non-relativistic Thomas-Fermi atom.

The first step is to establish, within the Møller box, the equilibrium condition for the electron distribution in an atom, the nucleus of which is at rest at $\mathbf{r}_n = 0$. The local Fermi momentum of a degenerate electron gas is

$$p_F(\mathbf{r}) = (3\pi^2)^{1/3}\hbar n^{1/3}(\mathbf{r}), \tag{4.25}$$

where $n(\mathbf{r})$ is the density of electrons. Thus, the electron charge density is

$$\varrho_e(\mathbf{r}) = -en(\mathbf{r}), \tag{4.26}$$

and the total charge density of the system is

$$\varrho(\mathbf{r}) = \varrho_e(\mathbf{r}) + Ze\,\delta(\mathbf{r}-\mathbf{r}_n)\,. \tag{4.27}$$

For a free atom at rest, the total Hamiltonian of the system then takes on the familiar form (cf. Gombás[14])

$$H = \int d^3r \left(mc^2 + \frac{3}{5}\frac{p_F^2(\mathbf{r})}{2m}\right) n(\mathbf{r}) + \int d^3r \frac{Ze\varrho_e(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_n|} + \frac{1}{2}\int d^3r \int d^3r' \frac{\varrho_e(\mathbf{r})\varrho_e(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{4.28}$$

where m is the electron mass.

For the total electric potential $\varphi(\mathbf{r})$, generated by the charge distribution (4.27), we have according to (3.15)

$$\varphi(\mathbf{r}) = \int d^3r' \frac{\varrho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\left(1 + \frac{x+x'}{2\lambda}\right). \tag{4.29}$$

The effective Lagrangian for the individual electron, moving in the total

potential φ, is given by (3.21), and the corresponding Hamiltonian for a single electron is to lowest order in v^2/c^2

$$H_e = \left(1+\frac{x}{\lambda}\right)\left(mc^2+\frac{\mathbf{p}^2}{2m}\right)-e\varphi(\mathbf{r}) . \tag{4.30}$$

All electron states up to the Fermi momentum $p_F(\mathbf{r})$ are occupied, and in equilibrium the maximum energy, E_{max}, that an electron can have at any point, is constant throughout the atom

$$\left(1+\frac{x}{\lambda}\right)(mc^2+\mu(\mathbf{r}))-e\varphi(\mathbf{r}) = E_{max} , \tag{4.31}$$

where we have introduced the potential

$$\mu(\mathbf{r}) = \frac{p_F^2(\mathbf{r})}{2m} = (3\pi^2)^{2/3}\frac{\hbar^2}{2m}n^{2/3}(\mathbf{r}) . \tag{4.32}$$

Combining (4.32) with the generalized Poisson equation (3.14), one may obtain the Thomas-Fermi equation in the Møller box.

In order to derive the reaction force on the nucleus due to the electrons, we note that from the potential (4.29), one obtains as a generalization of the relation (3.17)

$$\int d^3r\,\varrho(\mathbf{r})\frac{\partial\varphi(\mathbf{r})}{\partial x} = \frac{1}{2\lambda}\int d^3r'\,\frac{\varrho(\mathbf{r})\varrho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} . \tag{4.33}$$

This relation is valid for any static charge distribution and may in particular be applied to the charge distribution $\varrho_n(\mathbf{r})$ of the nucleus and the potential φ_n that it generates

$$\int d^3r\,\varrho_n(\mathbf{r})\frac{\partial\varphi_n(\mathbf{r})}{\partial x} = \frac{1}{2\lambda}\int d^3r\int d^3r'\,\frac{\varrho_n(\mathbf{r})\varrho_n(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} . \tag{4.34}$$

Next, we write the total potential φ in (4.33) as

$$\varphi(\mathbf{r}) = \varphi_n(\mathbf{r})+\varphi_e(\mathbf{r}), \tag{4.35}$$

where $\varphi_e(\mathbf{r})$ is the potential generated by electron charge distribution $\varrho_e(\mathbf{r})$. Subtracting (4.34) from (4.33) we obtain

$$\int d^3r\,\varrho_n(\mathbf{r})\frac{\partial\varphi_e(\mathbf{r})}{\partial x}+\int d^3r\,\varrho_e(\mathbf{r})\frac{\partial\varphi(\mathbf{r})}{\partial x} = \tag{4.36}$$

$$= \frac{1}{\lambda}\int d^3r\int d^3r'\,\frac{\varrho_n(\mathbf{r}')\varrho_e(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}+\frac{1}{2\lambda}\int d^3r\int d^3r'\,\frac{\varrho_e(\mathbf{r})\varrho_e(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} .$$

In this relation we may approximate the charge distribution $\varrho_n(\mathbf{r})$ of the nucleus by a delta function in accordance with (4.27). For the reaction force on the nucleus from the electron distribution we thus get

$$F_x = -Ze\frac{\partial \varphi_e(\mathbf{r})}{\partial x}\Big|_{\mathbf{r}_n=0}$$

$$= \int d^3r\, \varrho_e(\mathbf{r})\frac{\partial \varphi(\mathbf{r})}{\partial x} - \frac{1}{\lambda}\int d^3r\frac{Ze\varrho_e(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_n|} - \frac{1}{2\lambda}\int d^3r\int d^3r'\frac{\varrho_e(\mathbf{r})\varrho_e(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{4.37}$$

Expressing the potential φ in terms of μ, cf. the equilibrium condition (4.31), and using (4.32), we find

$$\int d^3r\varrho_e(\mathbf{r})\frac{\partial \varphi}{\partial x} = -\int d^3r\; n(\mathbf{r})\frac{\partial}{\partial x}\left\{\left(1+\frac{x}{\lambda}\right)(\mu(\mathbf{r})+mc^2)\right\}$$

$$= -\int d^3r\left\{n(\mathbf{r})\frac{1}{\lambda}\left(\mu(\mathbf{r})+mc^2\right) + (1+\frac{x}{\lambda})\;\frac{1}{3\pi^2}\left(\frac{2m}{\hbar^2}\mu\right)^{3/2}\frac{\partial \mu}{\partial x}\right\}$$

$$= -\frac{1}{\lambda}\int d^3r\; n(\mathbf{r})\left(mc^2+\frac{3}{5}\mu(\mathbf{r})\right), \tag{4.38}$$

where the last equation follows by partial integration. Inserting this expression in (4.37), the reaction force, F_x, becomes:

$$F_x = -\frac{H}{c^2}g_0 = -\left(Nm-\frac{B}{c^2}\right)g_0, \tag{4.39}$$

where H is the Thomas-Fermi Hamiltonian (4.28) and N the total number of electrons. For the total force required to keep the atom at rest, we thus again obtain the expression (4.5), where B is the binding energy of the Thomas-Fermi atom.

§ 5. Conclusions and General Outlook

In the previous chapters we have verified that there is equivalence between inertial mass and self-energy. The study was performed in considerable detail, including electrostatic interactions and kinetic energies, for hydrogen-like systems and the Thomas-Fermi model, within both classical mechanics and relativistic quantum mechanics. Moreover, there was detailed equivalence, i.e. equivalence for each term and for each element of the interaction energy. It was not necessary for the treatment that the system were stable. Without doubt, these are satisfactory results since they imply that all terms of a calculation of self-energies have a separate and simple significance.

The equivalence could be made specific in terms of the basic equation of motion (2.12) for a charged, composite system, $\mathbf{g}_e M = q\mathbf{E}_{ext}(\mathbf{r}_c, t)$. Thus, not only did the mass M contain detailed equivalence, but also the system, and its acceleration, could be represented by one point: the centre of charge $\mathbf{r}_c$. Higher order terms, like radiation damping, may afterwards be built into the above equation of motion. In connection with these results we showed that there is an error in the standard Born-Heitler calculation of self-mass from total self-force. As to the conventional formulae for self-momentum and self-energy, i.e. (1.8) and (1.9), we found that they resulted from an unwarranted variation of a constant term in the Lagrangian (2.15), and therefore could not be compared with the proper momenta and energies. It was apparent that if one kept to a Lagrangian formulation in describing a system, the undesirable expressions (1.8) and (1.9) were avoided, the need for Poincaré stresses did not arise, and detailed equivalence was explicit.

There is a more general background to our work, concerned with the consistency and aim of the description. As promised in the brief introductory remarks in § 1, we shall now discuss this background.

We have been concerned with composite systems, and with their primary property, i.e. their mass. It was supposed that we can speak consistently about such systems. But already in the wording *composite* systems it is implicit that a simpler concept exists. In point of fact, we have an idealized concept, that of a *particle*, sometimes referred to as an *elementary* particle, or a *point* particle. From old, a particle is conceived as an unchangeable building stone of matter. On the one hand, we then visualize a composite system as a swarm of particles interacting with each other. On the other hand, we have to compare the properties of this swarm with the properties of one particle, asking for the likeness between the two, as well as for their difference in behaviour.

For the purpose of this comparison, consider a composite system, be it a molecule, a liquid drop, a crystal, or an atomic nucleus, and note the following. If we act upon the system by means of comparatively weak forces, the forces varying sufficiently slowly in space and time, then the behaviour of the system will be as if it were a particle. This means that it has a certain mass, charge, inner angular momentum, magnetic dipole moment, etc. It can possibly be represented as a point in space as was shown in the equation of motion (2.12), just in the way a particle – if we are cautious – can possibly be described as a point in space. By acting on the system with such moderate forces, we can measure the properties of the system, properties which are conserved when the system remains isolated. In this comparison to a point particle we need not require that the system be absolutely stable when isolated. We can allow it to be unstable,

like a uranium nucleus with a probability of fissioning, or like a liquid drop which may evaporate. In such cases we can think of it as having conservation within sufficiently short time intervals, or with a certain width of its energy. Note in this connection that, in the main, it is permissible to use classical mechanics as well as quantum theory in the description of the system, although, of course, quantum mechanics will give a more precise account of the physical properties.

Thus, in the limiting case of weak and slowly varying external fields, we find that we must be able to describe a composite system and a particle in a like manner. It lies near at hand to demand that we are also able *to account for* their properties in a like manner. In a way, this hypothesis corresponds to the historical development of particle physics where successively, molecules, atoms, and atomic nuclei, etc., have been described as elementary particles. But it is more essential that actual calculations of basic properties of systems comply with our demand, in so far as we are able to calculate these properties. Correspondingly, the problem of equivalence of mass and energy must be our primary concern.

Consider then calculations of self-energies and self-masses for, on the one hand, composite systems, and, on the other hand, particles. In the case of composite systems this calculation is prescribed: we treat its constituents, e.g., electrons and atomic nuclei, as elementary particles, and only their interactions and their motion contribute to the additional self-energy and mass. It is important to notice that constituents of a composite system – constituents like the above atomic nuclei – often can be regarded as composite systems themselves, and so the division into constituents can be somewhat free. This possibility of a variable division into constituents leads to the further expectation that each separate interaction contribution, or kinetic energy contribution, should show equivalence. We described this as the demand of *detailed* equivalence, and we verified that it is fulfilled.

If we demand a systematic account, the above ought to be compared with self-energies for particles, such as the self-energy of an electron. The latter concept is not quite simple, however, and that mainly on three counts. First, the basic method of finding self-energies belongs primarily to composite systems, and we can merely maintain that the proper procedure for a supposedly elementary particle must not be in discord with the former. Second, the leading term in the electron self-energy is apparently divergent, whereas the physically observable parts of the self-energy, like the Lamb effect, appear only in higher order terms in expansions in powers of $1/c$. But our primary concern, for composite systems, was not to evaluate cumbersome higher order terms. Third, there is an interesting complication because of the spin and magnetic moment of the electron; in order to make a comparison, one must first analyse composite systems with spin, or inner angular momentum. We have made this study and found that a classical

system with spin must be described by at least two points, the centre of motion and the centre of charge. An account of these questions will be given in a separate publication.

Acknowledgments

This paper has been under way during one decade. The contents of it, and its aim, has changed considerably in that period. Part of the subject, systems with spin, has been reserved for a future publication. We are grateful to many friends for debates on equivalence as well as subjects akin to it. We have, in particular, profited much from discussions with E. Eilertsen, P. Kristensen, Vibeke Nielsen, W. J. Swiatecki, and A. Winther.

We are especially indebted to Lise Madsen for competent and careful preparation of the paper.

Appendix A

Electrodynamics in Møller Box

The line element (3.6) in the Møller box corresponds to a diagonal metric tensor g_{ik}. If we put $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$, we have

$$g_{00} = \left(1 + \frac{x}{\lambda}\right)^2,$$

$$g_{11} = g_{22} = g_{33} = -1, \tag{A1}$$

$$\sqrt{-g} = 1 + \frac{x}{\lambda},$$

where g is the determinant of g_{ik}.

The field equations are derived from the action principle

$$\delta S_f + \delta S_{int} = 0, \tag{A2}$$

where, with general covariant notation, the contributions to the action from the field and the interaction are, respectively,

$$S_f = \frac{1}{16\pi c}\int d^4x \sqrt{-g}\, F^{ik} F_{ik} \tag{A3}$$

and

$$S_{int} = -\frac{1}{c^2}\int d^4x \sqrt{-g}\, j^i \mathcal{A}_i. \tag{A4}$$

Here the field tensor is given in terms of the four-potentials by

$$F_{ik} = \frac{\partial}{\partial x^i}\mathcal{A}_k - \frac{\partial}{\partial x^k}\mathcal{A}_i, \tag{A5}$$

and hence the action S_f is invariant with respect to the gauge transformation

$$\mathcal{A}_i \rightarrow \mathcal{A}_i + \frac{\partial}{\partial x^i}\Lambda(\mathbf{r}, t), \tag{A6}$$

where $\Lambda(\mathbf{r}, t)$ is an arbitrary scalar function of the coordinates.

The condition that also the action S_{int} be invariant against the transformation (A6) yields the conservation law

$$\frac{\partial}{\partial x^i}(\sqrt{-g}\, j^i) = 0. \tag{A7}$$

Therefore, we introduce the conserved current $(\varrho c, \mathbf{s})$ as

$$j^i = \frac{1}{\sqrt{-g}}(\varrho c, \mathbf{s}), \tag{A8}$$

so that the total charge

$$q = \int d^3 r \, \varrho(\mathbf{r}, t) \tag{A9}$$

is a constant.

If a point charge q_1 is located at the position $\mathbf{r}_1(t)$, the current density takes the form

$$j^i = \frac{q_1}{\sqrt{-g}} \frac{dx^i}{dt} \delta(\mathbf{r} - \mathbf{r}_1(t)) . \tag{A10}$$

From (A4) and (A8) we have

$$S_{int} = -\int dt \int d^3 r \left(\varrho \mathcal{A}_0 + \frac{1}{c} s^\lambda \mathcal{A}_\lambda \right) \tag{A11}$$

where $\lambda = 1, 2, 3$.

Since the space integral in (A11) represents the interaction energy, we must introduce the scalar potential φ and the vector potential $\mathbf{A}$ through the relation

$$(\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) = (\varphi, -A_x, -A_y, -A_z) , \tag{A12}$$

so that

$$S_{int} = -\int dt \int d^3 r \left(\varrho \varphi - \frac{1}{c} \mathbf{s} \cdot \mathbf{A} \right). \tag{A13}$$

From eq. (A5) follows the homogeneous Maxwell equations

$$\frac{\partial}{\partial x^i} F_{kl} + \frac{\partial}{\partial x^k} F_{li} + \frac{\partial}{\partial x^l} F_{ik} = 0 , \tag{A14}$$

whereas the inhomogeneous equations are obtained from the action principle (A2), when the variation is carried out for fixed charge and current distributions. The result is (cf. ref. 12)

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^k} (\sqrt{-g} \, F^{ik}) = -\frac{4\pi}{c} j^i . \tag{A15}$$

The conservation law (A7) is of course one consequence of the Maxwell equations (A15).

The physical significance of the potentials φ and $\mathbf{A}$ is expressed by the equations of motion for a charged particle in an external field. These equations are found from the action principle

$$\delta S_{kin} + \delta S_{int} = 0 , \tag{A16}$$

where a variation of the particle coordinates is carried out for fixed potentials.

Here

$$S_{kin} = -mc^2 \int dt \left\{ \left(1 + \frac{x}{\lambda}\right)^2 - \frac{\mathbf{v}^2}{c^2} \right\}^{1/2}, \tag{A17}$$

$$S_{int} = -\int dt\, q \left\{ \varphi(\mathbf{r}(t)) - \frac{\mathbf{v}}{c} \mathbf{A}(\mathbf{r}(t)) \right\}, \tag{A18}$$

where m is the mass of the particle and q its charge. The corresponding Lagrangian is

$$L = -mc^2 \left\{ \left(1 + \frac{x}{\lambda}\right)^2 - \frac{\mathbf{v}^2}{c^2} \right\}^{1/2} - q\varphi(\mathbf{r}, t) + \frac{q}{c} \mathbf{v} \cdot \mathbf{A}(\mathbf{r}, t), \tag{A19}$$

and hence the equations of motion become

$$\frac{d}{dt} \frac{m\mathbf{v}}{\{(1+x/\lambda)^2 - \mathbf{v}^2/c^2\}^{1/2}} = \frac{-m(1+x/\lambda)\mathbf{g}_0}{\{(1+x/\lambda)^2 - \mathbf{v}^2/c^2\}^{1/2}} + q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right), \tag{A20}$$

where we have introduced the electromagnetic fields

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \tag{A21}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{A22}$$

In terms of these quantities the field action (A3) takes the simple form

$$S_f = \frac{1}{8\pi} \int dt \int d^3r \left(\frac{\mathbf{E}^2}{1+x/\lambda} - (1+x/\lambda)\mathbf{B}^2 \right). \tag{A23}$$

It can be convenient to express the Maxwell equations (A14) and (A15) in the following three-dimensional notation

$$\begin{aligned} &\text{div } \mathbf{B} = 0 \quad , \\ &\text{rot } \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad , \\ &\text{div } \mathbf{D} = 4\pi\varrho \, , \\ &\text{rot } \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c} \mathbf{s} \, , \end{aligned} \tag{A24}$$

where we have introduced the abbreviations

$$\mathbf{D} = \frac{1}{1+x/\lambda} \mathbf{E}, \tag{A25}$$

$$\mathbf{H} = \left(1 + \frac{x}{\lambda}\right) \mathbf{B}. \tag{A26}$$

Appendix B

Static Potentials in Møller Box

In this appendix we derive the potential generated by a charge at rest in the box. By introducing (A21) and (A25) into (A24), one obtains the generalized Poisson equation for the potential φ,

$$\Delta\varphi - \frac{1}{\lambda + x}\frac{\partial\varphi}{\partial x} = -4\pi q_1 \left(1 + \frac{x}{\lambda}\right)\delta(\mathbf{r} - \mathbf{r}_1)\,, \tag{B1}$$

where the charge q_1 is at rest at the position $\mathbf{r}_1$. In the text we only use the solution of this equation to first order in $1/\lambda$ as given by (B12). But for the sake of completeness we here give the exact solution of (B1), subject to the boundary condition that it vanishes at infinity in the box. This solution is a somewhat complicated analytical expression, which can, however, be given a simple geometrical representation.

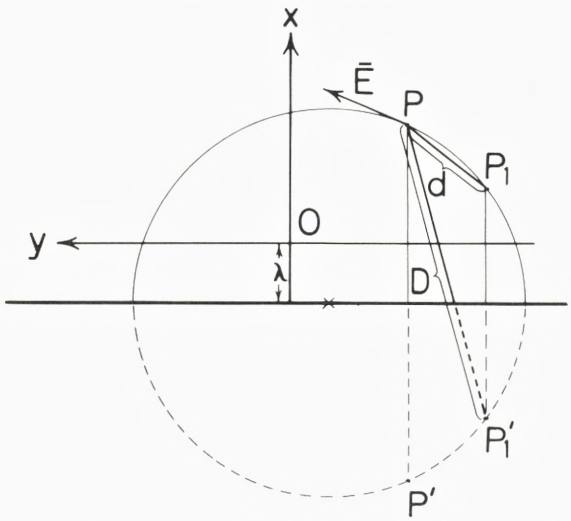

Fig. 4. Geometrical construction of potential and field from fixed charge in the Møller box. The x-axis in the box is chosen in the direction of $\mathbf{g}_0$. The thick line, at $x = -\lambda$, represents the bottom of the box. i.e. the region below this line does not belong to the box. The circle is drawn through the charge point P_1 and the field point P as well as through their mirror images, P_1' and P' beyond the bottom of the box. The field strength $\mathbf{E}$ is tangential to the circle at the point P.

Let P_1 represent the point in which the charge is located and P the field point (see Fig. 4). We draw a circle through these two points and their mirror images, P_1' and P', beyond the bottom of the box, at $x = -\lambda$. Denote the vectors from P_1 to P and from P_1' to P by $\mathbf{d}$ and $\mathbf{D}$, respectively. We shall prove that the solution of (B1) may be written

$$\varphi(\mathbf{r};\mathbf{r}_1) = \frac{q_1}{2\lambda}\left(\frac{d}{D} + \frac{D}{d}\right), \tag{B2}$$

where $d = |\mathbf{d}|$ and $D = |\mathbf{D}|$.

The right hand side of (B2) tends to the constant q_1/λ at infinity and at the bottom of the box. It is seen from Fig. 4 that d and D are given by

$$d = \{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2\}^{1/2} \tag{B3}$$

$$D = \{(x+\lambda+x_1+\lambda)^2 + (y-y_1)^2 + (z-z_1)^2\}^{1/2}. \tag{B4}$$

In order to show that (B2) is a solution of (B1), we observe that the Laplacian acting on d/D becomes

$$\Delta\frac{d}{D} = \frac{1}{D}\cdot\Delta d + d\Delta\frac{1}{D} + 2(\nabla d)\cdot\left(\nabla\frac{1}{D}\right) = -4\pi d\,\delta(\mathbf{D}) + \frac{1}{dD} - \frac{d}{D^3} + \frac{4(x_1+\lambda)^2}{dD^3}, \tag{B5}$$

and similarly

$$\Delta\frac{D}{d} = -4\pi D\,\delta(\mathbf{d}) + \frac{1}{dD} - \frac{D}{d^3} - \frac{4(x_1+\lambda)^2}{Dd^3}. \tag{B6}$$

Furthermore,

$$\begin{aligned}\frac{1}{\lambda+x}\frac{\partial}{\partial x}\frac{d}{D} &= \frac{1}{\lambda+x}\left(\frac{x-x_1}{dD} - d\,\frac{x+\lambda+x_1+\lambda}{D^3}\right),\\ \frac{1}{\lambda+x}\frac{\partial}{\partial x}\frac{D}{d} &= \frac{1}{\lambda+x}\left(\frac{x+\lambda+x_1+\lambda}{dD} - D\,\frac{(x+\lambda)-(x_1+\lambda)}{d^3}\right).\end{aligned} \tag{B7}$$

Finally, making use of the relation

$$D^2 - d^2 = 4(x+\lambda)(x_1+\lambda), \tag{B8}$$

we obtain

$$\left(\Delta - \frac{1}{\lambda+x}\frac{\partial}{\partial x}\right)\frac{q_1}{2\lambda}\left(\frac{d}{D}+\frac{D}{d}\right) = -4\pi\frac{q_1}{2\lambda}\left(D\,\delta(\mathbf{d}) + d\,\delta(\mathbf{D})\right). \tag{B9}$$

Since $\delta(\mathbf{D})$ vanishes everywhere within the box, we have verified that, apart from an additive constant, (B2) is the solution of (B1) with the desired boundary condition. It is easily seen that the electric field $\mathbf{E}$ at the point P is tangential to the circle and of magnitude

$$E(P) = q_1\left(1+\frac{x_1}{\lambda}\right)\left(\frac{1}{d^2} - \frac{1}{D^2}\right). \tag{B10}$$

Incidentally it may be remarked that a light ray, sent from the point P_1 to the point P, travels along the circle shown on the figure.

In order to obtain the potential to first order in $1/\lambda$ we expand as follows

$$D = 2\lambda + x_1 + x \ldots . \tag{B11}$$

Hence the potential becomes

$$\varphi \simeq \frac{q_1}{2\lambda}\frac{D}{d} \simeq \frac{q_1}{|\mathbf{r}-\mathbf{r}_1|}\left(1+\frac{x+x_1}{2\lambda}\right), \tag{B12}$$

which is the form used in the text.

The above result for the potential may also be derived by transforming the potentials, generated by a charge in hyperbolic motion, from the inertial frame to the Møller box and performing a gauge transformation. The potentials in the inertial frame were originally derived by Born[10].

Appendix C

Klein-Gordon Equation in Møller Box

Let K(t) denote the particular inertial frame which coincides with the Møller box at time t. In this frame, the Klein-Gordon equation for a spinless particle of charge q and mass m in an external potential $\mathcal{A}$ may be written as

$$\left(P_l - \frac{q}{c}\mathcal{A}_l\right)\left(P^l - \frac{q}{c}\mathcal{A}^l\right)\psi_{K(t)}(T) = m^2c^2\psi_{K(t)}(T)\,, \tag{C1}$$

where $\psi_{K(t)}(T)$, the wave function in the frame K(t), is a scalar quantity. With the identification

$$P_l = i\hbar\frac{\partial}{\partial X^l}\,, \tag{C2}$$

and imposing the Lorentz condition we obtain

$$\left(-\hbar^2\,\Box - 2\frac{q}{c}i\hbar\,\mathcal{A}^l\frac{\partial}{\partial X^l} + \frac{q^2}{c^2}\,\mathcal{A}^l\mathcal{A}_l\right)\psi_{K(t)}(T) = m^2c^2\psi_{K(t)}(T). \tag{C3}$$

We now notice that the wave function in the Møller box $\psi(t)$ is equal to the wave function $\psi_{K(t)}(T)$

$$\psi(t) = \psi_{K(t)}(T)\,. \tag{C4}$$

Moreover, since $\psi_{K(t)}$ is a scalar, the product $\mathcal{A}^l\,\partial\psi_{K(t)}/\partial X^l$ is an invariant. Therefore the Klein-Gordon equation in the Møller box is simply obtained by expressing this invariant and the d'Alembertian in non-Euclidean coordinates. For the latter operator we have the general expression (cf. ref. 12, §86)

$$\Box = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\sqrt{-g}\,g^{ik}\frac{\partial}{\partial x^k}\,. \tag{C5}$$

With the metric (A1) in the Møller box, this operator becomes

$$\begin{aligned}\Box &= \frac{1}{(1+x/\lambda)^2}\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \left(\Delta + \frac{1}{\lambda+x}\frac{\partial}{\partial x}\right)\\ &\simeq (1-2x/\lambda)\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \left(\Delta + \frac{1}{\lambda}\frac{\partial}{\partial x}\right),\end{aligned} \tag{C6}$$

where the last expression is valid to first order in $1/\lambda$. We shall only consider the case of a static potential in the Møller box, i.e. $\mathcal{A} = (\varphi,0,0,0)$. Hence we obtain

$$\mathcal{A}^l\frac{\partial}{\partial x^l} = g^{00}\mathcal{A}_0\frac{\partial}{\partial x^0} = \frac{1}{(1+x/\lambda)^2}\varphi\frac{1}{c}\frac{\partial}{\partial t} \simeq (1-2x/\lambda)\varphi\frac{1}{c}\frac{\partial}{\partial t}\,. \tag{C7}$$

Similarly,

$$\mathcal{A}^i \mathcal{A}_i = g^{00} \mathcal{A}_0^2 \simeq (1 - 2x/\lambda)\varphi^2 . \tag{C8}$$

Multiplying eq. (C3) from the left with the factor $(1+2x/\lambda)$, we obtain to first order in $1/\lambda$

$$\left\{-\hbar^2 \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{2q}{c} i\hbar\varphi \frac{1}{c}\frac{\partial}{\partial t} + \frac{q^2}{c^2}\varphi^2 + \left(1+2\frac{x}{\lambda}\right)\left(\Delta + \frac{1}{\lambda}\frac{\partial}{\partial x}\right)\right\}\psi = \left(1+2\frac{x}{\lambda}\right) m^2 c^2 \psi, \tag{C9}$$

or

$$\left(i\hbar\frac{\partial}{\partial t} - q\varphi\right)^2 \psi = \left\{m^2 c^4 \left(1+2\frac{x}{\lambda}\right) - \left(1+2\frac{x}{\lambda}\right)\hbar^2 \Delta - \frac{\hbar^2}{\lambda}\frac{\partial}{\partial x}\right\}\psi . \tag{C10}$$

It is convenient to rewrite this equation in a more symmetric form by introducing an auxiliary function

$$\Psi = \frac{1}{(1+x/\lambda)^{1/2}}\,\psi . \tag{C11}$$

Thereby we get from (10), again to first order in $1/\lambda$,

$$\left(i\hbar\frac{\partial}{\partial t} - q\varphi\right)^2 \Psi = \left\{m^2 c^4 \left(1+2\frac{x}{\lambda}\right) + c\mathbf{p}\left(1+2\frac{x}{\lambda}\right)c\mathbf{p}\right\}\Psi , \tag{C12}$$

where $\mathbf{p} = -i\hbar\boldsymbol{\nabla}$. We note that to this order eq. (C12) may be written

$$\left(i\hbar\frac{\partial}{\partial t} - q\varphi\right)^2 \Psi = H_0^2 \Psi , \tag{C13}$$

where H_0 is given by (4.6).

To the Klein-Gordon equation belongs a conserved four-current density $(\varrho, \mathbf{s})$. According to (C11) it is to first order in $1/\lambda$ given by

$$\begin{aligned} \varrho &= \frac{i\hbar}{2mc^2}\left(\Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t}\right) - \frac{q\varphi}{mc^2}|\Psi|^2 , \\ \mathbf{s} &= -\frac{i\hbar}{2m}\left(\Psi^*\left(1+2\frac{x}{\lambda}\right)\boldsymbol{\nabla}\Psi - \Psi\left(1+2\frac{x}{\lambda}\right)\boldsymbol{\nabla}\Psi^*\right). \end{aligned} \tag{C14}$$

These quantities obey the continuity equation $\partial\varrho/\partial t = -\operatorname{div}\mathbf{s}$ because of (C12).

Consider a hydrogen-like system, where a particle of charge q_2 moves in the potential φ given by (3.15). The demonstration of equivalence in this case differs slightly from the derivations in §4 for two reasons. First, the wave function belonging to a stationary state of the atom is not an eigenstate of a Hamiltonian as in the cases studied in the text. Second, the conserved density is not $|\Psi^2|$, but is given by (C14).

In a stationary state we have

$$\Psi(\mathbf{x},t) = \Psi(\mathbf{x})e^{\frac{-i}{\hbar}Et}. \tag{C15}$$

Thus the density ϱ takes the form

$$\varrho = \frac{E - q_2\varphi}{mc^2}|\Psi|^2, \tag{C16}$$

where Ψ satisfies the equations (C12) or (C13) with $i\hbar\partial/\partial t$ replaced by E.

We assume that the integral of the density ϱ, in (C16), is normalized to unity, and thus the charge density of the particle is equal to $q_2\varrho(\mathbf{r}_2)$. Hence we get for the expectation value, F_x, of the reaction force from the particle on the charge centre q_1 situated at the position $\mathbf{r}_1 = 0$

$$\begin{aligned} F_x &= -q_1\overline{\frac{\partial\varphi(\mathbf{r}_1;\mathbf{r}_2)}{\partial x_1}}\bigg|_{\mathbf{r}_1=0} \equiv -\int d^3r_2\, q_1\varrho(\mathbf{r}_2)\left(\frac{\partial\varphi(\mathbf{r}_1;\mathbf{r}_2)}{\partial x_1}\right) \\ &= -\int d^3r_2\,\varrho(\mathbf{r}_2)q_2\left(\frac{1}{\lambda}\varphi(\mathbf{r}_2;\mathbf{r}_1) - \frac{\partial\varphi(\mathbf{r}_2;\mathbf{r}_1)}{\partial x_2}\right). \end{aligned} \tag{C17}$$

Noticing the relationships

$$\begin{aligned} \int d^3r_2\varrho q_2\frac{\partial\varphi}{\partial x_2} &= -\int d^3r_2|\Psi|^2\frac{\partial}{\partial x_2}\frac{(E-q_2\varphi)^2}{2m_2c^2} \\ &= -\int d^3r_2\Psi^*\left[\frac{\partial}{\partial x_2}, \frac{H_0^2}{2m_2c^2}\right]\Psi \\ &= -\frac{1}{\lambda}\int d^3r_2\Psi^*\frac{(E-q_2\varphi)^2}{m_2c^2}\Psi \\ &= -\frac{1}{\lambda}\int d^3r_2\varrho(\mathbf{r}_2)(E-q_2\varphi), \end{aligned} \tag{C18}$$

we obtain equivalence in (C17)

$$F_x = -\frac{1}{\lambda}E\int d^3r_2\varrho(\mathbf{r}_2) = -\frac{E}{c^2}g_0. \tag{C19}$$

References

1. Jackson, J. D., Classical Electrodynamics, Chapter 17, Wiley, New York (1962).
2. Einstein, A., Ann. d. Phys. *17*, 891 (1905), Ann. d. Phys. *18*, 639 (1905). Reprinted in English translation in "The Principle of Relativity", Dover Publications (1952).
3. v. Laue, M., Inertia and Energy, in "Albert Einstein, Philosopher-Scientist", (Ed. A. Schilpp) The Library of Living Philosophers, Evanstone, Illinois (1949).
4. Jammer, M., Concepts of Mass, Chapter 11, Harvard University Press (1961).
5. Lorentz, H. A., The Theory of Electrons, Teubner Verlag, Leipzig (1916). Reprinted in Dover Publications (1952).
6. Pais, A., Developments in the Theory of the Electron. Institute for Advanced Studies and Princeton University (1948).
7. Feynman, R. P., Lectures on Physics II (Chapt. 28), Addison-Wesley, Reading, Mass. (1964).
8. Becker, R. and Sauter, F., Theorie der Elektrizität, Teubner Verlag, Stuttgart (1957).
9. Rohrlich, F., Am. J. Phys. *28*, 634 (1960), Ibid. *38*, 1310 (1970), and Classical Charged Particles, Addison-Wesley, Reading, Mass. (1965).
10. Born, M., Ann. d. Phys. *30*, 1 and 840 (1909).
11. Heitler, W., The Quantum Theory of Radiation, Oxford University Press, Oxford (1954).
12. Landau, L. and Lifshitz, I. M., Classical Theory of Fields, 3. ed. Pergamon Press, Oxford, New York (1973)
13. Møller, C., The Theory of Relativity, Oxford University Press, Oxford (1972).
14. Gombás, P., Die Statistische Theorie des Atoms und ihre Anwendungen, Springer Verlag, Wien (1969).

Indleveret til Selskabet december 1981.
Færdig fra trykkeriet juni 1982